

Quantum motion on a torus as a submanifold problem in a generalized Dirac's theory of second-class constraints

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A generalization of the Dirac's canonical quantization theory for a system with second-class constraints is proposed as the fundamental commutation relations that are constituted by all commutators between positions, momenta and Hamiltonian so they are simultaneously quantized in a self-consistent manner, rather than by those between merely positions and momenta so the theory either contains redundant freedoms or conflicts with experiments. The application of the generalized theory to quantum motion on a torus leads to two remarkable results: i) The theory formulated purely on the torus, i.e., based on the so-called the purely intrinsic geometry, conflicts with itself. So it provides an explanation why an intrinsic examination of quantum motion on torus within the Schrödinger formalism is improper. ii) An extrinsic examination of the torus as a submanifold in three dimensional flat space turns out to be self-consistent and the resultant momenta and Hamiltonian are satisfactory all around.

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I. INTRODUCTION

The embedding problem of quantum motion of a particle on a two-dimensional curved surface Σ^2 in the flat space R^3 has attracted much attention, including theoretical explorations [1–9] and experimental investigations [10, 11]. Fundamentally, there are two formalisms to investigate the quantum motion on Σ^2 . One is within the Schrödinger formalism that needs a wave function and another is within the Dirac one that purely deals with operators, but they usually give different predictions. In this section, we will mainly review these two formalisms, and present a generalization of the Dirac's canonical quantization theory for a system of the second-class constraints.

A. Schrödinger and Dirac formalism: discrepancies in curvature dependent quantum potentials

By the *Schrödinger formalism* we mean that the Schrödinger equation is first formulated in R^3 , actually in a curved shell of an equal and finite thickness δ whose intermediate surface coincides with the prescribed one Σ^2 (or equivalently, the particle moves within the range of the same width δ due to a confining potential around the surface), and an effective Schrödinger equation on the curved surface Σ^2 is then derived by taking the squeezing limit $\delta \rightarrow 0$ to confine the particle to the Σ^2 [1–3, 6]. It leads to a unique form of the so-called geometric potential [6, 10]

$$V_g = -\frac{\hbar^2}{2m} (M^2 - K) \quad (1)$$

that depends on both the mean and the gaussian curvature M and K which are, respectively, the extrinsic and the intrinsic curvature. This amounts to an extrinsic examination of the quantum motion on Σ^2 within the Schrödinger formalism. The potential (1) has been experimentally confirmed [10, 11]. To note that the extrinsic curvature M is a geometric consequence of embedding the system on Σ^2 in R^3 and is inaccessible with purely intrinsic description. However, for this formalism, we do not know why the Schrödinger equation can not be entirely formulated on Σ^2 without considering any embedding. We are familiar with a fact an intrinsic examination of the quantum motion on Σ^2 within the Schrödinger formalism that predicts no curvature dependent quantum potential, which is contrary to the experiments [10, 11].

By the *Dirac formalism* we mean to use the Dirac's canonical quantization theory on systems with the second-class constraints [12, 13], with an understanding that Dirac formalism can also be applied to the system that is considered

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either within purely intrinsic geometry on Σ^2 or as a submanifold in R^3 , predicting a curvature dependent potential V_D with two real parameters α and β [7, 8],

$$V_D = -\frac{\hbar^2}{2m} (\alpha M^2 - \beta K). \quad (2)$$

This form of the potential (2) can also be easily constructed by dimensional analysis for two geometric invariants M and K have dimension of $length^{-1}$ and $length^{-2}$, respectively. In comparison with the Schrödinger formalism, we have one more unknown associated with the Dirac one, that is, once taking the Σ^2 as a submanifold in R^3 we do not know what form of the potential can be singled out among a family of it (2). However, Schrödinger's theory gives an unambiguous choice with $\alpha = \beta = 1$ [1, 2, 4, 6].

So far, we find that both formalisms suffer from shortcomings. Since the extrinsic examination of the torus within the Schrödinger formalism has experimental supports, an immediate question is whether there is a possible theoretical framework from which we can fix the parameters α and β within a possibly generalized Dirac's theory, so rendering it compatible with Schrödinger's and also the experimental results. This question will be partially answered in this paper.

B. Schrödinger and Dirac formalism: discrepancies in momentum operators

In addition to the unique form of the geometric potential $V_g = -\hbar^2 (M^2 - K) / 2m$, Schrödinger's theory also leads to a unique definition of the geometric momentum \mathbf{p} [5, 6],

$$\mathbf{p} = -i\hbar(\mathbf{r}^\mu \partial_\mu + M\mathbf{n}), \quad (3)$$

where $\mathbf{r} = (x(x^1, x^2), y(x^1, x^2), z(x^1, x^2))$ is the position vector in R^3 on the surface Σ^2 whose local coordinates are $x^\mu \equiv (x^1, x^2)$ and $\mathbf{r}^\mu = g^{\mu\nu} \mathbf{r}_\nu = g^{\mu\nu} \partial \mathbf{r} / x^\nu$, and at this point \mathbf{r} , $\mathbf{n} = (n_x, n_y, n_z)$ denotes the normal and $M\mathbf{n}$ symbolizes the mean curvature vector field, another geometric invariant. Throughout the paper, the Einstein summation convention over repeated indices is used.

However, the present formulation of Dirac's theory opens a wide door to permit various definitions of the generalized momenta, including i) the well-known generalized ones $p_\mu = -i\hbar(\partial_\mu + \Gamma_\mu/2)$ which satisfy quantum commutator $[x^\nu, p_\mu] = i\hbar\delta_\mu^\nu$, where Γ_μ is the once-contracted Christoffel symbol $\Gamma_{\mu\nu}^\sigma$ constructed with Riemannian metric $g^{\mu\nu}$ [7], where greek letters μ, ν, σ , etc. run between 1 to 2, and ii) geometric momentum (3), and etc. [6, 8]. *It is very important to note that in the extrinsic examination of quantum motion on Σ^2 in R^3 , the local coordinates $x^\mu \equiv (x^1, x^2)$ are no longer position operators but parameters, and the position operators are $\mathbf{r} = (x(x^1, x^2), y(x^1, x^2), z(x^1, x^2))$.*

A framework based on the purely intrinsic geometry implies that every quantity solely relies on the Riemannian metric $g^{\mu\nu}$ and its various constructions such as Christoffel symbol $\Gamma_{\mu\nu}^\sigma$ and the gaussian curvature K . Consequently, neither momentum nor Hamiltonian in quantum mechanics depends on the extrinsic curvature. When the curvature dependent potential with (2) $\alpha \neq 0$ and geometric momentum (3) appear in a formulation of quantum mechanics for a system on Σ^2 , we in fact take the system under study to be embedded in R^3 , which is beyond the purely intrinsic geometry.

C. A generalization Dirac's theory for a system of the second-class constraints

We are deeply impressed by the very success of Schrödinger's theory that produces unique result of the geometric potential (2) and momentum (3), and also by the disturbing arbitrariness associated with Dirac's theory of the second-class constraints. As we know, Dirac's theory postulates that a quantum commutator $[A, B]$ of two variables A and B in quantum mechanics is achieved by direct correspondence of the Dirac's brackets $\{A, B\}_D$ as $\{A, B\}_D \rightarrow [A, B]$ which is defined by $[A, B] = i\hbar O(\{A, B\}_D)$ where $O(F)$ is used to emphasize the operator form of the classical quantity F in order to avoid possible confusion. When all constraints are removed, the Dirac bracket $\{A, B\}_D$ assumes its usual form, the Poisson bracket $\{A, B\}$. However, Dirac himself states that *fundamental commutation relations involve only those between canonical positions x_i and canonical momenta p_i* [12, 13].

One can ask a curious question: when there is no constraint, why there is no such a *fundamental* canonical quantization rule between $f (= x_i, p_i)$ and the Hamiltonian H as $[f, H] = i\hbar O(\{f, H\})$? This is because the direct quantization $[f, H] = i\hbar O(\{f, H\})$ might be redundant, or meaningless, or practically useless, etc. For instance, when the system has a classical analogue, the Hamiltonian is the same function of the positions and momenta in the quantum theory as in the classical theory, provided that the Cartesian system of axes is used [13–15]. In this case the rule $[f, H] = i\hbar O(\{f, H\})$ turns out to be redundant. When a quantum Hamiltonian has no classical analogy,

the canonical quantization rule $[f, H] = i\hbar O(\{f, H\})$ is meaningless. In many other cases, e.g., to quantize a classical Hamiltonian $H = \gamma x^3 p^3$ with γ being a real parameter, the rule should be imposed but is practically useless. Thus, it appears unacceptable to include the canonical quantization rule $[f, H] = i\hbar O(\{f, H\})$ as a fundamental element of a theory.

For systems with the second-class constraints, the situation is totally different. Discrepancies between either curvature dependent quantum potentials or momentum operators present when different formalisms, or different geometric points, are utilized. It strongly implies that, while the quantization of the system is performed, the proper operator form of positions, momenta and Hamiltonian are simultaneously determined in a self-consistent way. Therefore we have attempted to generalize the Dirac's theory so as to add $[f, H]$ into the category of the *fundamental commutation relations* which should also be directly achieved via following quantization rule [6],

$$[f, H] = i\hbar O(\{f, H\}_D), \quad f = x_i \text{ and } p_j. \quad (4)$$

In rest part of the paper, the convention $O(F) = F$ in quantum mechanics assumes without no longer emphasizing it an operator with the symbol O . These commutation relations (4) may not be applicable when the system has no constraint. So we would like to call them the second category of fundamental ones [6], whereas the existing ones between positions and momenta, the first.

This generalized Dirac's theory reproduces the usual form for the system that has a classical analogue but has not a constraint, together with the necessary utilization of the Cartesian system of axes, therefore enriches the Dirac formalism of quantum mechanics. We will call it the *general theory of the canonical quantization* (GTCQ).

D. Purpose and organization of the paper

As an application of the GTCQ to quantum motion on a sphere [6], we find that, on one hand, an attempt of trying a proper description within the purely intrinsic geometry proves problematic, and on the other hand, an account of embedding the sphere in three-dimensional space is very coherent. Notice that the classification theorem for compact surfaces states that [16], every compact orientable surface is homeomorphic either to a sphere or to a connected sum of tori, implying that if there is any difficulty associated with quantum mechanics for a particle constrained on a sphere or a torus, enormous theoretical problems would arise from dealing with an arbitrary two-dimensional curved surface in quantum mechanics. It forms one of the reasons that the sphere [6] and the torus [17–20] are used to test various theories. The main purpose of the present study is to take the torus to show that Dirac formalism is complementary to the Schrödinger one. The former eliminates the purely intrinsic description, and the latter gives the unique form of the geometric potential, while both define the identical form of the geometric momentum.

This paper is organized as follows. In following section II, we present the GTCQ for quantum motion on the torus within purely intrinsic geometry. Results show that the theory can never be consistently set up. In section III, we revisit the same problem as a submanifold in flat space R^3 with the GTCQ. Results show that the theory turns out to be self-consistent all around, and the obtained geometric momentum (3) and potential (1) are also satisfactory. Section IV briefly remarks and concludes this study.

II. GTCQ FOR A TORUS WITHIN INTRINSIC GEOMETRY

The toroidal surface is with two local coordinates $\theta \in [0, 2\pi), \varphi \in [0, 2\pi)$

$$\mathbf{r} = ((a + r \sin \theta) \cos \varphi, (a + r \sin \theta) \sin \varphi, r \cos \theta), \quad a > r \neq 0, \quad (5)$$

where φ is the azimuthal angle and θ the polar angle, and a and r are the outer and inner radii of the torus, respectively. The constraint is $r = b \neq 0$. In this section, we will first give the classical mechanics for motion on the torus, and then turn into the Dirac formalism of quantum mechanics. In classical mechanics, the theory appears nothing surprising, but after transition to quantum mechanics, it becomes contradictory to itself.

A. Classical mechanical treatment

The Lagrangian L in the toric coordinate system is,

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + (a + r \sin \theta)^2\dot{\varphi}^2) - \lambda(r - b), \quad (6)$$

where λ is the Lagrangian multiplier enforcing the constrained motion on the surface. The Lagrangian is singular because it does not contain the "velocity" $\dot{\lambda}$. Hence we need the Dirac formalism of the classical mechanics for a system with the second-class constraints, which gives the canonical momenta conjugate to r, θ, φ and λ in the following,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad (7)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad (8)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m(a + r \sin \theta)^2 \dot{\varphi}, \quad (9)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (10)$$

Eq. (10) represents the primary constraint:

$$\varphi_1 \equiv p_\lambda \approx 0, \quad (11)$$

hereafter symbol " \approx " implies a weak equality [13]. After all calculations are finished, the weak equality takes back the strong one. By the Legendre transformation, the primary Hamiltonian H_p is [13],

$$H_p = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{(a + r \sin \theta)^2}) + \lambda(r - b) + \dot{\lambda}p_\lambda, \quad (12)$$

where $\dot{\lambda}$ is also a Lagrangian multiplier guaranteeing that this Hamiltonian is defined on the symplectic manifold. The secondary constraints (not confusing with second-class constraints) are generated successively, then determined by the conservation condition [13],

$$\varphi_{i+1} \equiv \{\varphi_i, H_p\} \approx 0, \quad (i = 1, 2, \dots), \quad (13)$$

where $\{f, g\}$ is the Poisson bracket with $q_1 = r, q_2 = \theta, q_3 = \varphi$, and $p_1 = p_r, p_2 = p_\theta, p_3 = p_\varphi$,

$$\{f, g\} \equiv \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} + \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial p_\lambda} - (\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} + \frac{\partial f}{\partial p_\lambda} \frac{\partial g}{\partial \lambda}). \quad (14)$$

The complete set of the secondary constraints is,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = -(r - b) \approx 0, \quad (15)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = -\frac{p_r}{m} \approx 0, \quad (16)$$

$$\varphi_4 \equiv \{\varphi_3, H_p\} = \frac{\lambda}{m} - \frac{1}{m^2}(\frac{p_\theta^2}{r^3} + \frac{p_\varphi^2 \sin \theta}{(a + r \sin \theta)^3}) \approx 0, \quad (17)$$

$$\varphi_5 \equiv \{\varphi_4, H_p\} = \frac{\dot{\lambda}}{m} - \frac{3ap_\theta p_\varphi^2 \cos \theta}{m^3 r^2 (a + r \sin \theta)^4} \approx 0. \quad (18)$$

Eqs. (15) and (16) show, respectively, that on the surface of torus $r = b$, no motion along the normal direction is possible $p_r = 0$, while Eqs. (17) and (18) determine, respectively, the Lagrangian multipliers λ and $\dot{\lambda}$.

The Dirac bracket instead of the Poisson one for two variables A and B is defined by,

$$\{A, B\}_D \equiv \{A, B\} - \{A, \varphi_u\} C_{uv}^{-1} \{\varphi_v, B\}, \quad (19)$$

where the 4×4 matrix $C \equiv \{C_{uv}\}$ whose elements are defined by $C_{uv} \equiv \{\varphi_u, \varphi_v\}$ with $u, v = 1, 2, 3, 4$ from Eqs. (11) and (15)-(17). The inverse matrix C^{-1} is,

$$C^{-1} = \begin{pmatrix} 0 & C_{12}^{-1} & 0 & m \\ -C_{12}^{-1} & 0 & -m & 0 \\ 0 & m & 0 & 0 \\ -m & 0 & 0 & 0 \end{pmatrix}, \quad (20)$$

where

$$C_{12}^{-1} = \frac{3}{m} \left(\frac{p_\theta^2}{b^4} + \frac{p_\varphi^2 \sin^2 \theta}{(a + b \sin \theta)^4} \right). \quad (21)$$

Thus, the generalized positions q^μ ($= \theta, \varphi$) and momenta p_μ satisfy the following Dirac brackets,

$$\{q^\mu, q^\nu\}_D = 0, \quad \{p_\mu, p_\nu\}_D = 0, \quad \{q^\mu, p_\nu\}_D = \delta_\nu^\mu. \quad (22)$$

By use of the equation of motion,

$$\dot{f} = \{f, H_c\}_D, \quad (23)$$

we obtain those for the positions θ, φ and the momenta p_θ, p_φ , respectively,

$$\dot{\theta} \equiv \{\theta, H_c\}_D = \frac{p_\theta}{mb^2}, \quad \dot{\varphi} \equiv \{\varphi, H_c\}_D = \frac{p_\varphi}{m(a + b \sin \theta)^2}, \quad (24)$$

$$\dot{p}_\theta \equiv \{p_\theta, H_c\}_D = \frac{b \cos \theta p_\varphi^2}{m(a + b \sin \theta)^3}, \quad \dot{p}_\varphi \equiv \{p_\varphi, H_c\}_D = 0. \quad (25)$$

In these calculations (24) and (25), we in fact need only the usual form of Hamiltonian, $H_p \rightarrow H_c$,

$$H_c = \frac{1}{2m} \left(\frac{p_\theta^2}{b^2} + \frac{p_\varphi^2}{(a + b \sin \theta)^2} \right). \quad (26)$$

So far, the classical mechanics for the motion on the torus is complete and coherent in itself.

B. Quantum mechanical treatment

In quantum mechanics, we assume that the Hamiltonian takes the following general form,

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} [\nabla^2 + (\alpha M^2 - \beta K)] \\ &= -\frac{\hbar^2}{2m} \left[\frac{1}{b^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{b(a + b \sin \theta)} \frac{\partial}{\partial \theta} + \frac{1}{(a + b \sin \theta)^2} \frac{\partial^2}{\partial \varphi^2} \right. \\ &\quad \left. + \alpha \frac{1}{4} \left(\frac{a + 2b \sin \theta}{ab + b^2 \sin \theta} \right)^2 - \beta \frac{\sin \theta}{ab + b^2 \sin \theta} \right], \end{aligned} \quad (27)$$

where,

$$M = -\frac{1}{2} \frac{a + 2b \sin \theta}{ab + b^2 \sin \theta}, \quad K = \frac{\sin \theta}{ab + b^2 \sin \theta}.$$

We are ready to construct commutator $[A, B]$ of two variables A and B in quantum mechanics, which can be straightforwardly realized by a direct correspondence of the Dirac's brackets as $\{A, B\}_D \rightarrow [A, B]/i\hbar$. From the Dirac's brackets (22), the first category of the fundamental commutators between operators q^μ and p_ν are given by,

$$[q^\mu, q^\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [q^\mu, p_\nu] = i\hbar \delta_\nu^\mu. \quad (28)$$

In light of the GTCQ, we have the second category of fundamental commutators between q^μ and H from Eq. (24),

$$[\theta, H] = \frac{\hbar^2}{mb^2} \left(\frac{\partial}{\partial \theta} + \frac{b \cos \theta}{2(a + b \sin \theta)} \right) = i\hbar \frac{p_\theta}{mb^2}, \quad (29)$$

$$[\varphi, H] = \frac{\hbar^2}{m(a + b \sin \theta)^2} \frac{\partial}{\partial \varphi} = i\hbar \frac{p_\varphi}{m(a + b \sin \theta)^2}. \quad (30)$$

From these quantum commutators, the operators p_θ and p_φ are, respectively,

$$p_\theta = -i\hbar \left[\frac{\partial}{\partial \theta} + \frac{b \cos \theta}{2(a + b \sin \theta)} \right], \quad p_\varphi = -i\hbar \frac{\partial}{\partial \varphi}. \quad (31)$$

Using these operators, we can directly calculate two quantum commutators $[p_\theta, H]$ and $[p_\varphi, H]$ with quantum Hamiltonian (27), and the results are, respectively,

$$[p_\theta, H] = i\hbar \frac{b \cos \theta}{m(a + b \sin \theta)^3} p_\varphi^2 + i\hbar \frac{\hbar^2 \cos \theta (a^2(\alpha - 2\beta + 1) + 2ab(\alpha - \beta) \sin \theta - b^2)}{4bm(a + b \sin \theta)^3}, \quad (32)$$

$$[p_\varphi, H] = 0. \quad (33)$$

The second equation (33) is satisfactory, whereas the first one (32) can hardly hold true. In the GTCQ, the quantum commutator $[p_\theta, H]$ (32) must be the canonical quantization of the Dirac bracket (25). We get, with noting the mutual commutability between two observables p_φ and θ ,

$$i\hbar \{p_\theta, H\}_D = \frac{i\hbar b \cos \theta p_\varphi^2}{m(a + b \sin \theta)^3}. \quad (34)$$

In comparison with the right-handed sides of the Eqs. (32) and (34), we obtain a unique solution,

$$\alpha = \beta = \frac{a^2 - b^2}{a^2} (\neq 1), \quad (35)$$

which leads an unacceptable curvature dependent quantum potential that includes the extrinsic curvature M ,

$$V_D = -\frac{\hbar^2}{2m} \frac{a^2 - b^2}{a^2} (M^2 - K) = -\frac{\hbar^2}{2m} \frac{a^2 - b^2}{4b^2 (a + b \sin \theta)^2}. \quad (36)$$

However, no matter what other values of α and β are chosen, there is a manifest breakdown of the canonical quantization rule between Dirac bracket $\{p_\theta, H\}_D$ (25) and the quantum commutator $[p_\theta, H]$ (32). So we see that the intrinsic geometry is insufficient for the GTCQ to be self-consistent.

If using original form of the Dirac's theory instead, we still have results (31)-(33) but we can never require them as the canonical quantization of the relevant Dirac brackets (24)-(25). It is sheer nonsense for we neither are able to exclude the extrinsic curvature M , nor give a unambiguous prediction of the curvature dependent potential to be testable by experiment.

One should be noted that we have not introduced additional assumptions such as "dummy factors" techniques [21] etc. in passing from Dirac's brackets to the quantum commutators. They mean further generalizations of the Dirac's theory.

In classical limit $\hbar \rightarrow 0$, all inconsistency vanishes, as expected.

C. Summary

From the studies in this section, we see that the GTCQ of second-class constraints for quantum motion on the torus can not be consistently formulated. We therefore need to invoke an extrinsic examination of the same problem, as will be done in next section.

III. GTCQ FOR A TORUS AS A SUBMANIFOLD

The surface equation of the torus (5) in Cartesian coordinates (x, y, z) is given by,

$$f(\mathbf{x}) \equiv a^2 - b^2 + (x^2 + y^2 + z^2) - 2a\sqrt{x^2 + y^2} = 0. \quad (37)$$

In this section, we will also first give the classical mechanics for motion on the torus within the Dirac formalism of the classical mechanics for a system with the second-class constraints, and then turn into quantum mechanics. The GTCQ proves to be self-consistent all around and the resultant momenta and Hamiltonian are exactly those given by the Eq. (3) and (1), respectively.

A. Classical mechanical treatment

The Lagrangian L in the Cartesian coordinate system is,

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \lambda f(\mathbf{x}). \quad (38)$$

The generalized momentum \mathbf{p} whose three components p_i ($i = x, y, z$) and p_λ canonically conjugate to variables x_i ($x_1 = x, x_2 = y, x_3 = z$) and λ , are given by, respectively,

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, (i = 1, 2, 3), \quad (39)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (40)$$

Eq. (40) represents the primary constraint,

$$\varphi_1 \equiv p_\lambda \approx 0. \quad (41)$$

By the Legendre transformation, the primary Hamiltonian H_p is,

$$H_p = \frac{1}{2m} p_i^2 + \lambda f(\mathbf{x}) + \dot{\lambda} p_\lambda. \quad (42)$$

The secondary constraints are determined by successive use of the Poisson brackets,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = -(a^2 - b^2 + x_i^2 - 2a\sqrt{x^2 + y^2}) \approx 0, \quad (43)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = -\frac{2\left(\sqrt{x^2 + y^2}(p_x x + p_y y + p_z z) - a(p_x x + p_y y)\right)}{m\sqrt{x^2 + y^2}} \approx 0, \quad (44)$$

$$\varphi_4 \equiv \{\varphi_3, H_p\} = \frac{4\lambda\left(a^2 - 2a\sqrt{x^2 + y^2} + x_i^2\right)}{m} + \frac{2a(p_y x - p_x y)^2}{m^2(x^2 + y^2)^{3/2}} - \frac{2p_i^2}{m^2} \approx 0, \quad (45)$$

$$\varphi_5 \equiv \{\varphi_4, H_p\} = \frac{4\dot{\lambda}\left(a^2 - 2a\sqrt{x^2 + y^2} + x_i^2\right)}{m} - \frac{6a(p_x x + p_y y)(p_y x - p_x y)^2}{m^3(x^2 + y^2)^{5/2}} \approx 0. \quad (46)$$

Similarly, the Dirac bracket between two variables A and B is defined by,

$$\{A, B\}_D = \{A, B\} - \{A, \varphi_u\} D_{uv}^{-1} \{\varphi_v, B\}, \quad (47)$$

where the 4×4 matrix $D \equiv \{D_{uv}\}$ whose elements are defined by $D_{uv} \equiv \{\varphi_u, \varphi_v\}$ with $u, v = 1, 2, 3, 4$ from Eqs. (41) and (43)-(45). The inverse matrix D^{-1} is easily carried out,

$$D^{-1} = \begin{pmatrix} 0 & D_{12}^{-1} & 0 & \kappa \\ -D_{12}^{-1} & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & 0 \\ -\kappa & 0 & 0 & 0 \end{pmatrix}, \quad (48)$$

where,

$$D_{12}^{-1} = \frac{\left(3a^2 - 7a\sqrt{x^2 + y^2}\right)(p_y x - p_x y)^2 + 4(x^2 + y^2)^2 p_i^2}{4b^4 m (x^2 + y^2)^2}, \quad \kappa = \frac{m}{4b^2}. \quad (49)$$

Then primary Hamiltonian H_p assumes its usual one: $H_p \rightarrow H_c$,

$$H_c = \frac{p_x^2 + p_y^2 + p_z^2}{2m}. \quad (50)$$

All fundamental Dirac's brackets are as follows,

$$\{x_i, x_j\}_D = 0, \quad (51)$$

$$\{x_i, p_j\}_D = \delta_{ij} - \frac{1}{b^2} f_i f_j, \quad (52)$$

$$\{p_i, p_j\}_D = -\frac{1}{b^2} \left[f_i \left(p_j + \frac{a(xp_y - yp_x)}{(x^2 + y^2)^{3/2}} (y\delta_{1j} - x\delta_{2j}) \right) - f_j \left(p_i + \frac{a(xp_y - yp_x)}{(x^2 + y^2)^{3/2}} (y\delta_{1i} - x\delta_{2i}) \right) \right], \quad (53)$$

$$\{x_i, H_c\}_D = \frac{p_i}{m} = \dot{x}_i, \quad (54)$$

$$\{p_i, H_c\}_D = -\frac{1}{mb^2} \left[f_i \left(p_x^2 + p_y^2 + p_z^2 - \frac{a(xp_y - yp_x)^2}{(x^2 + y^2)^{3/2}} \right) \right] = \dot{p}_i, \quad (55)$$

where $f_i = x_i - a(x\delta_{1i} + y\delta_{2i}) / \sqrt{x^2 + y^2}$.

B. Quantum mechanical treatment

Now let us turn to quantum mechanics. The first category of the fundamental commutators between operators x_i and p_i are, by quantization of (51)-(53),

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar \left(\delta_{ij} - \frac{1}{b^2} f_i f_j \right), \quad (56)$$

$$[p_i, p_j] = -\frac{i\hbar}{b^2} \left[f_i \left(p_j + a \frac{L_z (y\delta_{1j} - x\delta_{2j}) + (y\delta_{1j} - x\delta_{2j}) L_z}{2(x^2 + y^2)^{3/2}} \right) - f_j \left(p_i + a \frac{L_z (y\delta_{1i} - x\delta_{2i}) + (y\delta_{1i} - x\delta_{2i}) L_z}{2(x^2 + y^2)^{3/2}} \right) \right], \quad (57)$$

where $L_z = xp_y - yp_x$. It seems that we have complicated operator-ordering problem as passing from the Dirac bracket Eq. (53) to the quantum commutator (57). In fact, only one pair between the noncommuting observables x_i (precisely, $(y\delta_{1j} - x\delta_{2j})$) and L_z matters, and the product of $(y\delta_{1j} - x\delta_{2j})$ and L_z can be made Hermitian by a symmetric construction, $((y\delta_{1j} - x\delta_{2j}) L_z + L_z (y\delta_{1j} - x\delta_{2j})) / 2$. Other products of factors f_i (or f_j) and L_z impose no operator-ordering problem because of the Jacobi identity.

There is a family of the momenta p_i all of them are solutions to the Eq. (57), as explicitly shown in [8]. With these momenta p_i at hand, we completely do not know the correct form of the quantum Hamiltonian, as suggested by Eq. (50). It is therefore understandable that the quantum Hamiltonian would contain arbitrary parameters.

However, the GTCQ requires the second category of the fundamental commutators as $[x_i, H]$ and $[p_i, H]$. We immediately find that the momenta p_i from following commutators,

$$[x_i, H] = i\hbar \frac{p_i}{m}. \quad (58)$$

The obtained momenta p_i are nothing but three components of the geometric momentum (3) on the torus [20],

$$p_x = -i\hbar \left(\frac{\cos \theta \cos \varphi}{b} \frac{\partial}{\partial \theta} - \frac{\sin \varphi}{a + b \sin \theta} \frac{\partial}{\partial \varphi} - \frac{a + 2b \sin \theta}{2b(a + b \sin \theta)} \sin \theta \cos \varphi \right), \quad (59)$$

$$p_y = -i\hbar \left(\frac{\cos \theta \sin \varphi}{b} \frac{\partial}{\partial \theta} + \frac{\cos \varphi}{a + b \sin \theta} \frac{\partial}{\partial \varphi} - \frac{a + 2b \sin \theta}{2b(a + b \sin \theta)} \sin \theta \sin \varphi \right), \quad (60)$$

$$p_z = i\hbar \left(\frac{\sin \theta}{b} \frac{\partial}{\partial \theta} + \frac{a + 2b \sin \theta}{2b(a + b \sin \theta)} \cos \theta \right). \quad (61)$$

As to the form of quantum Hamiltonian, we also start from the general form (27), and now resort to the following

complicated operator-ordering arrangement with $w_{\pm} = (x \pm iy)^{3/2}$,

$$\begin{aligned}
[p_i, H] = & -\frac{i\hbar}{mb^2} \{mHf_i + mf_iH \\
& -\frac{a}{4}\alpha_1[f_i(L_z\frac{1}{w_+}L_z\frac{1}{w_-} + \frac{1}{w_-}L_z\frac{1}{w_+}L_z) + (L_z\frac{1}{w_+}L_z\frac{1}{w_-} + \frac{1}{w_-}L_z\frac{1}{w_+}L_z)f_i] \\
& -\frac{a}{4}\alpha_2[f_i(L_z\frac{1}{w_+}\frac{1}{w_-}L_z + \frac{1}{w_-}L_zL_z\frac{1}{w_+}) + (L_z\frac{1}{w_+}\frac{1}{w_-}L_z + \frac{1}{w_-}L_zL_z\frac{1}{w_+})f_i] \\
& -\frac{a}{4}\alpha_3[f_i(\frac{1}{w_+}L_zL_z\frac{1}{w_-} + L_z\frac{1}{w_-}\frac{1}{w_+}L_z) + (\frac{1}{w_+}L_zL_z\frac{1}{w_-} + L_z\frac{1}{w_-}\frac{1}{w_+}L_z)f_i] \\
& -\frac{a}{4}\alpha_4[f_i(\frac{1}{w_+}L_z\frac{1}{w_-}L_z + L_z\frac{1}{w_-}L_z\frac{1}{w_+}) + (\frac{1}{w_+}L_z\frac{1}{w_-}L_z + L_z\frac{1}{w_-}L_z\frac{1}{w_+})f_i] \\
& -\frac{a}{2}\alpha_5\frac{1}{w_+w_-}(f_iL_z^2 + L_z^2f_i)\},
\end{aligned} \tag{62}$$

where α_k , ($k = 1, 2, \dots, 5$) are five real parameters satisfying $\sum \alpha_k = 1$. In comparison of both sides of this equation, we find that the solution $\alpha = \beta = 1$, and two of the five real parameters α_k are freely to be specified,

$$\alpha_1 = \frac{11}{9} - \alpha_4 - \alpha_5, \alpha_2 = \alpha_3 = -\frac{1}{9}. \tag{63}$$

We see that free parameters remain, but they are irrelevant to observable quantities such as momentum and potential. In fact, with $\alpha = \beta = 1$ in (2), a much simpler choice of the operator-ordering without free parameters is possible,

$$\begin{aligned}
[p_i, H] = & -\frac{i\hbar}{mb^2} \{mHf_i + mf_iH + \frac{1}{9}\frac{a}{4}[(\frac{1}{w_+}f_iL_z^2\frac{1}{w_-} + \frac{1}{w_-}f_iL_z^2\frac{1}{w_+}) \\
& + (\frac{1}{w_+}L_z^2\frac{1}{w_-}f_i + \frac{1}{w_-}L_z^2\frac{1}{w_+}f_i)] - \frac{10}{9}\frac{a}{2}\frac{1}{w_+w_-}(f_iL_z^2 + L_z^2f_i)\}.
\end{aligned} \tag{64}$$

Even we can by no means exhaust all possible forms of the operator-ordering, from Eqs. (62) and (64), we can at least conclude that the curvature dependent potential (2) given by the Dirac formalism converges to the geometric potential (1) given by the Schrödinger one.

C. Summary

An examination of the motion on torus as a submanifold problem in GTCQ ensures a highly self-consistent description, and this formalism comes compatible with the Schrödinger one.

IV. REMARKS AND CONCLUSIONS

It is long known that Dirac's theory of second-class constraints, in which the fundamental commutation relations involve only those between canonical positions and canonical momenta, contains redundant freedoms and causes difficulty sometimes. To overcome the problems, we recently put forward a proposal that the commutators between the positions, momenta, and Hamiltonian form a full set of the fundamental commutation relations to construct a self-consistent quantum theory, the so-called GTCQ. Then the GTCQ produces a unique form of the geometric momentum, and imposes additional requirement on the form of the Hamiltonian via the curvature dependent potential that has no direct analogy. We see that the geometric potential comes as the consequence of the extrinsic examination of the constrained motion.

Through a careful analysis of the quantum motion on a torus, we demonstrate that the purely intrinsic geometry does not suffice for the GTCQ to be self-consistently formulated, but an extrinsic examination of the torus in three dimensional flat space does. Our study implies that the Dirac formalism is complementary to the Schrödinger one. The former can be helpful to eliminate the intrinsic description, and the latter gives the unique form of the geometric potential, while both define the identical form of the geometric momentum.

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